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DIFFERENTIABILITY WITH RESPECT TO (t, s) OF THE PARABOLIC EVOLUTION OPERATOR

BY
ALESSANDRA LUNARDI
Dipartimento di Matematica, Via Ospedale 72, 09124 Cagliari, Italy

ABSTRACT
We give further regularity results with respect to (t, s) for the evolution operator $G(t, s)$ of abstract parabolic initial value problems in general Banach space. Such results are then used to establish a representation formula for the solutions of parabolic initial-boundary value problems with nonvanishing data at the boundary.

1. Introduction

The study of the evolution operator for abstract parabolic equations in general Banach space X began in the 1960's with the papers of Sobolevskii and Tanabe ([15], [16]), who studied initial value problems of the kind

$$u'(t) = A(t)u(t) + f(t), \quad s < t \leqq T; \quad u(s) = u_0 \tag{1.1}$$

under the assumptions that the linear operators $A(t)$ have the same domain D and generate analytic semigroups in X, and $t \rightarrow A(t)$ is Hölder continuous with values in $L(D, X)$. Further developments of the theory led one to consider the case of non-constant domains $D(A(t))$ (see e.g. [9], [20], [1], [5] and the references quoted there). All these papers are devoted to existence and estimates, in several norms, of $t \rightarrow G(t, s)$, and to the variation of constants formula, which gives (under suitable assumptions on u_0 and f) the solution of problem (1.1):

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$$(1.2)\qquad u(t)=G(t,s)u_0+\int_s^t G(t,\sigma)f(\sigma)d\sigma,\qquad s\leqq t\leqq T.$$

Not much seems to be known about further regularity properties of $G(t,s)$ (especially w.r.t. s), which is the subject of the present paper. Komatsu showed in [10] that, if $A(\cdot)A(0)^{-1}$ is analytically extendible to a complex sector around the positive real semi-axis, then $(t,s)\to G(t,s)$ is analytic for $t>s$. Gevrey regularity was studied by Tanabe in his book [17].

Here we consider the case of constant (but not necessarily dense) domain D, and we assume that $t\to A(t)$ belongs to $C^{1+\alpha}([0,T];L(D,X))$. Among the results, we quote the following: for every $x\in X$, $G(t,s)x$ is $C^{2+\alpha}$ with respect to t and $C^{1+\alpha-\varepsilon}$ with respect to s in the set $\{(t,s)\in\mathbf{R}^2,\ 0\leqq s<t\leqq T\}$ for each $\varepsilon>0$; moreover $G_s(t,s)x$ and $G_{tt}(t,s)x$ have singularities like $(t-s)^{-1}$ and $(t-s)^{-2}$, respectively, as t approaches s. Estimates for $G_s(t,s)x$, $G_{st}(t,s)x$ and $G_{tt}(t,s)x$ are given also for x belonging to some interpolation space between D and X; they turn out to be optimal, compared with the corresponding ones in the autonomous case $A(t)=A$. For deriving such estimates, we use the construction of the evolution operator of [14].

As an application, in Section 3 we consider a parabolic non-homogeneous initial-boundary value problem (here and in the following, repeated indices mean summation):

$$(1.3)\qquad \begin{cases} u_t(t,x)=a_{ij}(t,x)D_{ij}u(t,x)+b_i(t,x)D_iu(t,x)+c(t,x)u(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0<t\leqq T,\quad x\in\Omega, \\ u(0,x)=u_0(x),\quad x\in\Omega, \\ \mathscr{B}u(t,x)=g(t,x),\quad 0<t\leqq T,\quad x\in\partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and either $(\mathscr{B}f)(x)=f(x)$ (Dirichlet boundary condition) or $(\mathscr{B}f)(x)=\beta_i(x)D_if(x)+\gamma(x)f(x)$ (mixed non-tangential boundary condition). Under suitable assumptions on the data, we show that, in the case of the Dirichlet b.c., the solution of (1.3) admits the representation

$$(1.4)\qquad u(t,\cdot)=G(t,0)u_0+\int_0^t G_s(t,s)D(s)g(s,\cdot)ds,\qquad 0\leqq t\leqq T.$$

Here $G(t,s)$ is the evolution operator in the space $X=L^p(\Omega)$ generated by the family $A(t):D\to X$, $A(t)f=a_{ij}(t,\cdot)D_{ij}f+b_i(t,\cdot)D_if+c(t,\cdot)f$, $D=$

$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $D(s)$ is the so-called Dirichlet mapping at the time s, i.e. $D(s)\psi$ is the solution v of

$$(1.5) \qquad a_{ij}(s,\cdot)D_{ij}v + b_i(s,\cdot)D_i v + c(s,\cdot)v = 0 \quad \text{in } \Omega, \quad v = \psi \quad \text{in } \partial\Omega.$$

A similar representation formula holds in the case of the mixed boundary condition: we get

$$(1.6) \qquad u(t,\cdot) = G(t,0)u_0 + \int_0^t G_s(t,s)M(s)g(s,\cdot)ds, \qquad t \geqq 0$$

where, now, $G(t,s)$ is the evolution operator in the space $X = L^p(\Omega)$ generated by the family $A(t): E \to X$, $A(t)f = a_{ij}(t,\cdot)D_{ij}f + b_i(t,\cdot)D_i f + c(t,\cdot)f$,

$$E = \{f \in W^{2,p}(\Omega);\ \beta_i(x)D_i f(x) + \gamma(x)f(x) = 0 \text{ on } \partial\Omega\}$$

and $M(s)$ is the "mixed mapping" at the time s, i.e. $M(s)\psi$ is the solution z of

$$(1.7) \qquad \begin{aligned} &a_{ij}(s,\cdot)D_{ij}z + b_i(s,\cdot)D_i z + c(s,\cdot)z = 0 \qquad \text{in } \Omega,\\ &\beta_i(\cdot)D_i z + \gamma(\cdot)z = \psi \qquad \text{in } \partial\Omega. \end{aligned}$$

Formulas (1.4) and (1.6) are quite similar to the corresponding ones in the case where A does not depend on time, introduced by Balakrishnan in [7]; it is of interest in boundary control theory (see, e.g., the papers [11], [19], and the book [7], concerning the autonomous case). Other generalizations of the Balakrishnan formula to the time-dependent case may be found in [6], [2].

2. Further regularity results

Let X and D be Banach spaces, endowed with the norms $\| \ \|$, $\| \ \|_D$ respectively, and let $0 < \alpha < 1$, $T > 0$, $A(t): [0,T] \to L(D,X)$ be such that

$$(2.1) \qquad t \to A(t) \in C^{1+\alpha}([0,T]; L(D,X)),$$

(2.2) for every $t \in [0,T]$, $A(t)$ generates an analytic semigroup $e^{\sigma A(t)}$ in X,

(2.3) there is $c \geqq 1$ such that $c^{-1}\|x\|_D \leqq \|x\| + \|A(t)x\| \leqq c\|x\|_D$ for each $x \in D$, $0 \leqq t \leqq T$.

Then (see [14]) there is a family of evolution operators $G(t,s) \in L(X)$ such that

$$(2.4) \qquad G(t,s) = e^{(t-s)A(s)} + W(t,s), \qquad 0 \leqq s \leqq t \leqq T$$

where $t \to W(t,s)x$ is the unique solution w of

$$(2.5) \quad w'(t) = A(s)w(t) + [A(t) - A(s)](w(t) + e^{(t-s)A(s)}x), \ s < t \leqq T; \ w(s) = 0.$$

We shall use the representation formula (2.4) for studying further regularity properties of $G(t,s)$. With this aim we shall consider the family of real interpolation spaces

$$X_\theta = (X, D)_{\theta,\infty}, \qquad 0<\theta<1 \tag{2.6}$$

(see [18, ch. 1.14] for equivalent definitions and norms).

We also introduce a class of Banach spaces: if B is any Banach space and $a<b$, $0<\sigma<1$, $0\leqq\beta<1$, we set

$$\begin{aligned} Z_{\beta,\sigma}(a,b;B) = \Big\{ u:[a,b]\to B;\ & u\in C^\sigma([a+\varepsilon,b];B)\ \forall\varepsilon\in]0,b-a[, \\ \| u \|_{Z_{\beta,\sigma}(a,b;B)} = & \sup_{a<t<b} (t-a)^\beta \| u(t) \|_B \\ & + \sup_{0<\varepsilon<b-a} \varepsilon^{\beta+\sigma}[u]_{C^\sigma([a+\varepsilon,b];B)} < +\infty \Big\}. \end{aligned} \tag{2.7}$$

Such spaces are useful to describe the Hölder regularity properties of analytic semigroups (and parabolic evolution operators) up to $t=0$: for instance, if A generates an analytic semigroup e^{tA} in X, it is easy to see that the function $t\to e^{tA}x$ belongs to $Z_{0,\sigma}(0,T;X)\cap Z_{\theta,\sigma}(0,T;X_\theta)$ for every $\sigma,\theta\in]0,1[$ and $x\in X$.

2.1. *The function $e^{\sigma A(s)}x$*

For every $x\in X$ and $s\in[0,T]$, the function $\sigma\to e^{\sigma A(s)}x$ belongs obviously to $C^\infty(]0,+\infty[;D(A(s))^n)$ for every $n\in\mathbf{N}$, and

$$\partial^n/\partial\sigma^n(e^{\sigma A(s)}x) = (A(s))^n e^{\sigma A(s)}, \qquad 0\leqq s\leqq T,\quad \sigma>0,\quad x\in X.$$

In particular, thanks to (2.3), $\sigma\to e^{\sigma A(s)}x$ belongs to $C^\infty(]0,+\infty];D)$ and there are $M_1, M_2>0$ such that

$$\begin{aligned} &\| e^{\sigma A(s)}x \|_D \leqq \sigma^{-1}M_1 \| x \|, \quad \| A(s)e^{\sigma A(s)}x \|_D \leqq \sigma^{-2}M_2 \| x \|, \\ &\qquad\qquad \sigma>0,\quad 0\leqq s\leqq T,\quad x\in X. \end{aligned} \tag{2.8}$$

Moreover, for every $\theta\in]0,1[$ there are $M_{a,\theta}, M_{2,\theta}>0$ such that

$$\begin{aligned} &\| e^{\sigma A(s)}x \|_D \leqq \sigma^{\theta-1}M_{1,\theta} \| x \|_\theta, \quad \| A(s)e^{\sigma A(s)}x \|_D \leqq \sigma^{\theta-2}M_{2,\theta} \| x \|_\theta, \\ &\qquad\qquad \sigma>0,\quad 0\leqq s\leqq T,\quad x\in X_\theta. \end{aligned} \tag{2.9}$$

For every $x \in X$ and $\sigma > 0$, the function $s \to e^{\sigma A(s)}x$ belongs to $C^1(\,]0, T]; D)$, and we have

$$(2.10) \qquad \frac{\partial}{\partial s} e^{\sigma A(s)}x = -\frac{1}{2\pi i}\int_\gamma e^{\lambda\sigma}(\lambda - A(s))^{-1}A'(s)(\lambda - A(s))^{-1}x\, d\lambda$$

where γ is a suitable path in the complex plane joining $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some $\theta \in]\pi/2, \pi[$; γ may be chosen independent of s and such that

$$(2.11) \qquad \|(\lambda - A(s))^{-1}\|_{L(X)} \leqq M/|\lambda|, \qquad \|(\lambda - A(s))^{-1}\|_{L(X,D)} \leqq M$$

where M is some positive constant, independent of λ and s. From (2.10) and (2.11) it follows easily that there are $N_0, N_1 > 0$ such that

$$(2.12) \qquad \|\partial/\partial s\, e^{\sigma A(s)}x\| \leqq N_0\|x\|, \quad \|\partial/\partial s\, e^{\sigma A(s)}x\|_D \leqq \sigma^{-1}N_1\|x\|, \qquad \sigma > 0, \quad x \in X.$$

Using again (2.10) and (2.11) and recalling that A belongs to $C^{1+\alpha}([0, T]; L(D, X))$, we get also

$$(2.13) \qquad \left\|\frac{\partial}{\partial s} e^{tA(s)}x - \frac{\partial}{\partial s} e^{rA(s)}x\right\|_D \leqq N_2 \int_r^t \sigma^{-2} d\sigma\, \|x\|, \qquad 0 \leqq s \leqq T, \quad 0 < r \leqq T, \quad x \in X,$$

$$(2.14) \qquad \left\|\frac{\partial}{\partial s} e^{\sigma A(s)}x_{|s=s_1} - \frac{\partial}{\partial s} e^{\sigma A(s)}x_{|s=s_0}\right\| + \left\|\sigma\left(\frac{\partial}{\partial s} e^{\sigma A(s)}x_{|s=s_1} - \frac{\partial}{\partial s} e^{\sigma A(s)}x_{|s=s_0}\right)\right\|_D \leqq N_3(s_1 - s_0)^\alpha\|x\|, \quad 0 \leqq s_0 \leqq s_1 \leqq T, \quad \sigma > 0, \quad x \in X$$

where N_2, N_3 are positive constants.

2.2. The function $W(t, s)x$

In [14] we showed that $t \to W(t, s)x$ belongs to $Z_{1-\beta,\beta}(s, T; D)$, $t \to W_t(t, s)x$ belongs to $Z_{1-\beta,\beta}(s, T; X)$ for every $x \in X$ and $\beta \in]0, 1[$; we showed also that $s \to W(t, s)x$ is β-Hölder continuous for $s \leqq t - \varepsilon$, $\varepsilon > 0$, without giving precise estimates of its Hölder norm as $\varepsilon \to 0$. In this subsection we shall study some regularity properties of $W(t, s)x$ up to $t = s$. The main result is the following:

Proposition 2.1. *Let assumptions* (2.1), (2.2), (2.3) *hold. Then, for every*

$x \in X$ there exists $W_s(t,s)x$ for $0 \leqq s < t \leqq T$, $(s,t) \neq (0,T)$, and the following estimates hold:

$$\text{(2.15)}\quad \begin{aligned} &\text{(i)}\ \| W_s(t,s)x \| \leqq K_1 \| x \|, \ 0 \leqq s < t \leqq T, \ x \in X, \\ &\text{(ii)}\ \| W_s(t,s_1)x - W_s(t,s_0)x \| \leqq K_2(\sigma)(s_1 - s_0)^{\alpha\sigma}(t-s_1)^{-\alpha} \| x \|, \\ &\quad 0 \leqq s_0 \leqq s_1 < t \leqq T, \ x \in X \end{aligned}$$

for every $\sigma \in]0,1[$. Moreover, for each $\theta \in]0,1[$ there is $K_3(\theta) > 0$ such that

$$\text{(2.16)}\quad \begin{aligned} &\| W_s(t,s_1)x - W_s(t,s_0)x \| \leqq K_3(\theta)(s_1 - s_0)^{\alpha}(t-s_1)^{-\alpha} \| x \|_\theta, \\ &\qquad 0 \leqq s_0 \leqq s_1 < t \leqq T, \ x \in X_\theta. \end{aligned}$$ ■

For proving Proposition 2.1 we need some technical lemmas.

LEMMA 2.2. *Under assumptions* (2.1), (2.2), (2.3), *for every $x \in X$ and $\beta \in]0,1[$ the function $t \to W(t,s)x$ belongs to $Z_{0,\beta}(s,T;D)$ and $t \to W_t(t,s)x$ belongs to $Z_{0,\beta}(s,T;X)$. There is $C_1(\beta) > 0$ such that*

$$\text{(2.17)}\quad \| W(\cdot,s)x \|_{Z_{0,\beta}(s,T;D)} + \| W_t(\cdot,s)x \|_{Z_{0,\beta}(s,T;X)} \leqq C_1(\beta) \| x \|.$$

The proof is quite analogous to that of Proposition 2.2 of [14], the difference being that now A is Lipschitz continuous instead of only Hölder continuous. ■

COROLLARY 2.3. *Under assumptions* (2.1), (2.2), (2.3), *for every $\beta \in [0,1[$, $\sigma \in]0,1[$, and $f \in Z_{\beta,\sigma}(s,T;X)$, the function u given by formula* (1.2) (*with $u_0 = 0$*) *is the solution of problem* (1.1) (*with $u_0 = 0$*), *it belongs to $Z_{\beta,\sigma}(s,T;D)$, and u' belongs to $Z_{\beta,\sigma}(s,T;X)$. Moreover, there is $C_2(\beta,\sigma) > 0$ such that*

$$\text{(2.18)}\quad \| u \|_{Z_{\beta,\sigma}(s,T;D)} + \| u' \|_{Z_{\beta,\sigma}(s,T;X)} \leqq C_2(\beta,\sigma) \| f \|_{Z_{\beta,\sigma}(s,T;X)}.$$

The proof is the same as that of corollary 2.3(i) of [14]. ■

LEMMA 2.4. *Under assumptions* (2.1), (2.2), (2.3), *there is $C_3 > 0$ such that for every $x \in X$ and $0 \leqq s < t \leqq T$, $-s \leqq h \leqq T - t$ we have*

$$\text{(2.19)}\quad \begin{aligned} &\| W(t+h,s+h)x - W(t,s)x \|_D + \| W_t(t+h,s+h)x - W_t(t,s)x \| \\ &\leqq C_3 |h|^{\alpha}(t-s)^{-\alpha/2} \| x \|. \end{aligned}$$

PROOF. The function

$$v(t) = W(t+h,s+h)x - W(t,s)x, \qquad s \leqq t \leqq \Gamma$$

($\Gamma = T - h$ if $h \geqq 0$, $\Gamma = T$ if $h \leqq 0$) satisfies

$$\text{(2.20)}\quad v'(t) = A(t)v(t) + \phi_h(t) + \psi_h(t), \quad s < t \leqq \Gamma; \quad v(s) = 0$$

where

$$\phi_h(t) = [A(t+h) - A(t)]W(t+h, s+h)x, \quad s < t \leqq \Gamma,$$

$$\psi_h(t) = [A(t) - A(s)]e^{(t-s)A(s)}x - [A(t+h) - A(s+h)]e^{(t-s)A(s+h)}x, \quad s < t \leqq \Gamma.$$

Both ϕ_h and ψ_h are bounded in $[s, \Gamma]$ and Hölder continuous in $[s+\varepsilon, \Gamma]$: actually, for every $\beta \in]0, 1[$ we have, by (2.17) and (2.8), (2.12):

$$\| \phi_h(t) \| + \| \psi_h(t) \| \leqq \{ \| A' \|_\infty |h| C_1(\beta) + [A']_{C^\alpha} |h|^\alpha M_1 + \| A' \|_\infty N_1 |h| \} \| x \|$$

and, for $s < s + \varepsilon \leqq r < t \leqq \Gamma$:

$$\| \phi_h(t) - \phi_h(r) \| \leqq \{ [A']_{C^\alpha} |h| (t-r)^\alpha C_1(\beta) + \| A' \|_\infty |h| (t-r)^\beta (r-s)^{-\beta} C_1(\beta) \} \| x \|,$$

$$\| \psi_h(t) - \psi_h(r) \| \leqq \begin{cases} 2([A']_{C^\alpha} M_1 + \| A' \|_\infty N_1 T^{1-\alpha}) |h|^\alpha \| x \|; \\ 2([A']_{C^\alpha} (r-s)^{\alpha-1} M_1 + \| A' \|_\infty N_1) |h| \, \| x \|; \\ 2\left(\| A' \|_\infty (t-r)(r-s)^{-1} M_1 + \| A' \|_\infty (r-s) M_2 \int_{r-s}^{t-s} \sigma^{-2} d\sigma \right) \| x \| \\ \quad \leqq 2 \| A' \|_\infty (M_1 + M_2)(t-r)(r-s)^{-1} \| x \| \end{cases}$$

so that, for every $\sigma, \theta \in]0, 1[$ with $\sigma + \theta \leqq 1$, we have

$$\| \psi_h(t) - \psi_h(r) \| \leqq c(\sigma, \theta)(t-r)^\sigma |h|^{\theta + \alpha(1-\theta-\sigma)} \varepsilon^{-\sigma-(1-\alpha)\theta} \| x \|.$$

Choosing now $\sigma = (1-\alpha)/2$ and $\theta = \alpha/2$, we find that ψ_h belongs to $Z_{\beta,\sigma}(s, T; X)$, with $\beta = (\alpha - \alpha^2)/2$, and its $Z_{\beta,\sigma}$-norm is bounded by const $\cdot |h|^\alpha$. Applying Corollary 2.3 to problem (2.20) we get the statement. ■

PROOF OF PROPOSITION 2.1. In order to consider i.v. problems starting at $t = 0$, it is convenient to introduce the function

(2.21) $V(t, s)x = W(t+s, s)x, \quad 0 \leqq s \leqq T, \quad 0 \leqq t \leqq T - s, \quad x \in X.$

Once we have shown that V is differentiable w.r.t. both arguments for $0 < t < T - s$, we will get

(2.22) $W_s(t,s)x = -V_t(t-s,s)x + V_s(t-s,s)x, \quad s<t<T, \quad x\in X$

and estimates (2.15), (2.16) will follow easily from the corresponding ones concerning V_t and V_s.

The function $w(t) = W(t+s,s)x$ is the solution of

$$w'(t) = A(t+s)w(t) - [A(t+s)-A(s)]e^{tA(s)}x, \quad 0<t\leqq T-s; \quad w(0)=0 \tag{2.23}$$

and belongs to $Z_{0,\beta}(0,T-s;D)$, for every $\beta\in]0,1[$. Differentiating formally problem (2.23) w.r.t. s, we get an initial value problem for the unknown $z(t) = V_s(t,s)x$:

$$z'(t) = A(t+s)z(t) + A'(t+s)W(t+s,s)x - [A'(t+s)-A'(s)]e^{tA(s)}x - [A(t+s)-A(s)]\partial/\partial s(e^{tA(s)}x), \quad 0<t\leqq T-s; \quad z(0)=0. \tag{2.24}$$

We will show the following:

(a) Problem (2.24) has a unique solution $z^{(s)}\in Z_{1-\alpha,\alpha}(0,T-s;D)$, with

$$\| z^{(s)} \|_{Z_{1-\alpha,\alpha}(0,T-s;D)} + \| (d/dt)z^{(s)} \|_{Z_{1-\alpha,\alpha}(0,T-s;X)} \leqq C_4 \| x \| . \tag{2.25}$$

(b) $\lim_{h\to 0, h\in[-s,T-s]} h^{-1}[V(t,s+h)x - V(t,s)x] = z^{(s)}(t)$ for $0<t\leqq T-s$.

(c) For every $\sigma\in]0,1[$ there is $C_5(\sigma)>0$ such that

$$\| z^{(s)}(t) - z^{(r)}(t) \| \leqq C_5(\sigma)|s-r|^{\alpha\sigma} \| x \| , \quad x\in X \tag{2.26}$$

and for every $\theta\in]0,1[$ there is $C_6(\theta)>0$ such that

$$\| z^{(s)}(t) - z^{(r)}(t) \| \leqq C_6(\theta)|s-r|^{\alpha} \| x \|_\theta, \quad x\in X_\theta. \tag{2.27}$$

Proof of (a). We have only to show that the functions

$$\phi(t) = A'(t+s)W(t+s,s)x, \quad 0\leqq t\leqq T-s,$$

$$\psi(t) = [A'(t+s)-A'(s)]e^{tA(s)}x, \quad 0\leqq t\leqq T-s,$$

$$\chi(t) = [A(t+s)-A(s)]\partial/\partial s(e^{tA(s)}x), \quad 0\leqq t\leqq T-s$$

belong to $Z_{1-\alpha,\alpha}(0,T-s;X)$, and then to apply Corollary 2.3, since the family of operators $B(t) = A(t+s)$, $0\leqq t\leqq T-s$, satisfies assumptions (2.1), (2.2), (2.3). We have:

$$t^{1-\alpha}(\|\phi(t)\| + \|\psi(t)\| + \|\chi(t)\|)$$
$$\leqq t^{1-\alpha}(\|A'\|_x C_1(\alpha) + [A']_{C^\alpha} t^{\alpha-1} M_1 + \|A'\|_x N_1)\|x\|$$
$$\leqq \{T^{1-\alpha}\|A'\|_x (C_1(\alpha) + N_1) + [A']_{C^\alpha} M_1\}\|x\|$$

and, for $0 < \varepsilon \leqq r < t \leqq T - s$:

$$\varepsilon\|\phi(t) - \phi(r)\| \leqq \varepsilon[A']_{C^\alpha}(t-r)^\alpha C_1(\alpha)\|x\| + \varepsilon^{1-\alpha}\|A'\|_x (t-r)^\alpha C_1(\alpha)\|x\|$$
$$\leqq (T[A']_{C^\alpha} + T^{1-\alpha}\|A'\|_x) C_1(\alpha)(t-r)^\alpha \|x\|,$$
$$\varepsilon\|\psi(t) - \psi(r)\| \leqq \varepsilon(\|A'(t+s) - A'(r+s)\|_{L(D,X)}\|e^{tA(s)}x\|_D$$
$$+ \|A'(r+s) - A'(s)\|_{L(D,X)}\|e^{tA(s)}x - e^{rA(s)}x\|_D$$
$$\leqq (M_1 + M_2/\alpha)[A']_{C^\alpha}(t-r)^\alpha \|x\|,$$
$$\varepsilon\|\chi(t) - \chi(r)\| \leqq \varepsilon(\|A(t+s) - A(r+s)\|_{L(D,X)}\|\partial/\partial s(e^{tA(s)}x)\|_D$$
$$+ \|A(r+s) - A(s)\|_{L(D,X)}\|\partial/\partial s(e^{tA(s)}x - e^{rA(s)}x)\|_D)$$
$$\leqq (N_1 + N_2)\|A'\|_x (t-r)\|x\|.$$

Therefore $\phi + \psi + \chi$ belongs to $Z_{1-\alpha,\alpha}(0, T-s; X)$, and statement (a) holds thanks to Corollary 2.3.

Proof of (b). For $0 \leqq s < T$, $0 \leqq t_0 \leqq T - s$ ($(s, t_0) \neq (0, T)$) and $x \in X$, set

$$z_h(t) = h^{-1}(V(t, s+h)x - V(t,s)x)$$
$$= h^{-1}(W(t+s+h, s+h)x - W(t+s, s)x), \quad 0 \leqq t \leqq t_0.$$

If $s = 0$ and $t_0 \neq T$, z_h is defined for $0 < h \leqq T - t_0$; if $s > 0$, z_h is defined for $h \neq 0$, $-s \leqq h \leqq T - t_0 - s$. We want to show that, for each t_0, $z_h \to z$ as $h \to 0$, where z is the solution of (2.24). The function $t \to z_h(t) - z(t)$, $0 \leqq t \leqq t_0$, satisfies:

$$z_h'(t) - z'(t) = A(t+s)(z_h(t) - z(t)) + f_h(t), \quad 0 < t \leqq t_0; \quad (z_h - z)(0) = 0$$

where

$$f_h(t) = A(t+s)(z_h(t) - z(t))$$
$$+ \{h^{-1}[A(t+h+s) - A(t+s)] - A'(t+s)\}W(t+s, s)x$$
$$+ h^{-1}[A(t+h+s) - A(t+s)][W(t+h+s, h+s)x - W(t+s, s)x]$$
$$- \{h^{-1}[A(t+h+s) - A(t+s) - A(h+s) + A(s)]$$

$$- [A'(t+s) - A'(s)]\}e^{tA(s)}x$$
$$+ h^{-1}[A(t+h+s) - A(t+s) - A(h+s) + A(s)](e^{tA(h+s)}x - e^{tA(s)}x)$$
$$- [A(t+s) - A(s)][h^{-1}(e^{tA(h+s)}x - e^{tA(s)}x) - \partial/\partial s(e^{tA(s)}x)],$$
$$0 \leqq t \leqq t_0.$$

The linear operators $B(t) = A(t+s)$, $0 \leqq t \leqq t_0$, satisfy assumptions (2.1), (2.2), (2.3), and f_h belongs to $Z_{1-\alpha,\alpha}(0, t_0; X)$ for each h, so that $z_h - z$ may be represented by

$$z_h(t) - z(t) = \int_0^t U(t,\sigma) f_h(\sigma) d\sigma, \qquad 0 \leqq t \leqq t_0$$

where $U(t,\sigma)$ is the evolution operator generated by the family $\{B(t)\}$. We want to prove that $z_h(t) \to z(t)$ as $h \to 0$ for every $t \in [0, t_0]$. Since $U(t,\sigma)$ is bounded in $L(X)$, we have only to show that $f_h \to 0$ in $L^1(0, t_0; X)$.

We have:

$$\| \{h^{-1}[A(t+h+s) - A(t+s)] - A'(t+s)\} W(t+s, s)x \|$$
$$\leqq [A']_{C^\alpha} |h|^\alpha C_1(\alpha) \| x \| \quad \text{(by (2.17))};$$
$$\| h^{-1}[A(t+h+s) - A(t+s)][W(t+h+s, h+s)x - W(t+s, s)x] \|$$
$$\leqq \| A' \|_\infty C_3 t^{-\alpha/2} |h|^\alpha \| x \| \quad \text{(by (2.19))};$$
$$\| \{h^{-1}[A(t+h+s) - A(t+s) - A(h+s) + A(s)]$$
$$- [A'(t+s) - A'(s)]\} e^{tA(s)}x \|$$
$$\leqq \left\| \int_0^t [A'(t+s+\sigma h) - A'(t+s) - A'(s+\sigma h) + A'(s)] d\sigma \right\|_{L(D,X)} M_1 t^{-1} \| x \|$$
$$\leqq 2[A']_{C^\alpha} M_1 |h|^{\alpha/2} t^{\alpha/2 - 1} \| x \| ;$$
$$\| h^{-1}[A(t+h+s) - A(t+s) - A(h+s) + A(s)](e^{tA(h+s)}x - e^{tA(s)}x) \|$$
$$\leqq [A']_{C^\alpha} N_1 t^{\alpha-1} |h| \, \| x \| \quad \text{(by (2.12))};$$
$$\| [A(t+s) - A(s)][h^{-1}(e^{tA(h+s)}x - e^{tA(s)}x) - \partial/\partial s(e^{tA(s)}x)] \|$$
$$\leqq \| A' \|_\infty N_3 |h|^\alpha \| x \| \quad \text{(by (2.14))}.$$

Therefore, $f_h \to 0$ in $L^1(0, t_0; X)$ as $h \to 0$, and statement (b) is proved.

Proof of (c). Let $0 \leqq r < s \leqq T$. The function

$$w(t) = z^{(s)}(t) - z^{(r)}(t), \qquad 0 \leqq t \leqq T - r$$

is the solution of

$$w'(t) = A(t+r)w(t) + f(t), \quad 0 < t \leqq T - r, \quad w(0) = 0$$

where

$$\begin{aligned} f(t) = {} & [A(t+s) - A(t+r)]z_s(t) \\ & + [A'(t+s)W(t+s,s)x - A'(t+r)W(t+r,r)x] \\ & - \{[A'(t+s) - A'(s)]e^{tA(s)}x - [A'(t+r) - A'(r)]e^{tA(r)}x\} \\ & - \{[A(t+s) - A(s)]\partial/\partial s(e^{tA(s)}x) - [A(t+r) - A(r)]\partial/\partial r(e^{tA(r)}x)\}, \\ & \qquad 0 \leqq t \leqq T - r. \end{aligned}$$

Arguing as in in point (b), we find

$$\sup_{0 \leqq t \leqq T-r} \| w(t) \| \leqq \text{const} \cdot \| f \|_{L^1(0,T-r,X)}.$$

For every $\sigma \in]0, 1[$ and $t \in]0, T - r[$ we have, by (2.25), (2.17), (2.12), (2.14):

$$\begin{aligned} \| f(t) \| \leqq \{ & \| A' \|_\infty C_4 t^{\alpha-1}(s-r) + [A']_{C^\alpha}(s-r)^\alpha C_1(\alpha) \\ & + \| A' \|_\infty C_3 t^{-\alpha/2}(s-r)^\alpha \\ & + 2[A']_{C^\alpha} M_1 (s-r)^{\alpha\sigma} t^{\alpha(1-\sigma)-1} + [A']_{C^\alpha} N_1 t^{\alpha-1}(s-r) \\ & + [A']_{C^\alpha} N_1 (s-r)^\alpha + \| A' \|_\infty N_3 (s-r)^\alpha \} \| x \| . \end{aligned}$$

If x belongs to X_θ, then

$$\begin{aligned} & \| [A'(t+s) - A'(s)]e^{tA(s)}x - [A'(t+r) - A'(r)]e^{tA(r)}x \| \\ & \leqq [A']_{C^\alpha} M_{1,\theta}(s-r)^\alpha t^{\alpha-1} \| x \|_\theta + [A']_{C^\alpha} N_1 t^{\alpha-1}(s-r) \| x \| , \end{aligned}$$

the other estimates remaining unchanged. Point (c) is so proved.

We are now ready to show estimates (2.15) and (2.16). Points (a) and (b) imply that there exists $\partial/\partial s(V(t,s)x) = z^{(s)}(t)$ for $0 \leqq s < T$, $0 \leqq t \leqq T - s$ ($(s, t) \neq (0, T)$), and (2.22) holds. (2.15)(i) follows now from (2.17) and (2.25), recalling that $z' \in Z_{1-\alpha,\alpha}(0, T - s; X)$ implies that z is bounded with values in X. In fact, it is Hölder continuous, since we have

$$\| z(t) - z(r) \| = \left\| \int_r^t z'(\sigma) d\sigma \right\| \leqq \alpha^{-1}(t-r)^\alpha \sup_{0 < \sigma \leqq T-s} \| \sigma^{1-\alpha} z'(\sigma) \| .$$

Let us show (2.15)(ii). By (2.19) and (2.17) we have, for $0 \leqq s_0 < s_1 < t \leqq T$ and $0 < \sigma < 1$:

$$\begin{aligned} &\| V_t(t - s_1, s_1)x - V_t(t - s_0, s_0)x \| \\ &= \| W_t(t, s_1)x - W_t(t - s_1 + s_0, s_0)x \| \\ &\quad + \| W_t(t - s_1 + s_0, s_0)x - W_t(t, s_0)x \| \\ &\leqq C_3(t - s_1)^{-\alpha/2}(s_1 - s_0)^\alpha \| x \| + C_1(\alpha)(t - s_1)^{-\alpha}(s_1 - s_0)^\alpha \| x \| . \end{aligned}$$

(2.15)(ii) follows now from (2.25) and (2.26), whereas (2.15)(iii) follows from (2.25) and (2.27). ■

2.3. The function $G(t, s)$

Throughout the subsection, assumptions (2.1), (2.2), (2.3) are assumed to hold.

PROPOSITION 2.5. *For every $x \in X$, $G(t, s)x$ is differentiable w.r.t. s for $t > s$, and we have*

(2.28) (i) $G_s(t, s)x = -A(s)e^{(t-s)A(s)}x + \partial/\partial s(e^{\sigma A(s)}x)_{|\sigma = t-s} + W_s(t, s)x$, $0 \leqq s < t < T, x \in X$,
(ii) $G_s(t, s)x = G(t, r)G_s(r, s)x, 0 \leqq s < r < t < T, x \in X$.

Moreover, $G_s(t, s)x$ belongs to D for $t > s$ and it is differentiable w.r.t. t for $t > s$, with

(2.29) $$G_{st}(t, s)x = A(t)G_s(t, s)x, \quad 0 \leqq s < r < t < T, \quad x \in X.$$

If $x \in D$, we have also

(2.30) $$G_s(t, s) = -G(t, s)A(s)x, \qquad 0 \leqq s < t < T.$$

If $x \in D$ and $A(s)x$ belongs to the closure of D in X, then $G(t, s)x$ is differentiable w.r.t. s up to $t = s$, and (2.30) holds also for $t = s$.

There are $\gamma_1, \gamma_2, \gamma_1(\theta), \gamma_2(\theta) > 0$ such that

(2.31)
(i) $\| G_s(t, s)x \| \leqq \gamma_1(t - s)^{-1} \| x \|, x \in X$,
(ii) $\| G_s(t, s)x \| \leqq \gamma_1(\theta)(t - s)^{\theta - 1} \| x \|_\theta, x \in X_\theta$,
(iii) $\| G_s(t, s)x \| \leqq \gamma_1 \| x \|_D, x \in D$;

$$
(2.32)\quad
\begin{aligned}
&\text{(i)}\ \ \| G_{st}(t,s)x \| \leqq \gamma_2 (t-s)^{-2} \| x \|,\ x\in X,\\
&\text{(ii)}\ \ \| G_{st}(t,s)x \| \leqq \gamma_2(\theta)(t-s)^{\theta-2} \| x \|_\theta,\ x\in X_\theta,\\
&\text{(iii)}\ \ \| G_{st}(t,s)x \| \leqq \gamma_2 (t-s)^{-1} \| x \|_D,\ x\in D.
\end{aligned}
$$

For every $\sigma \in]0, 1[$ *there is* $\gamma_3(\sigma) > 0$ *such that*

$$
(2.33)\quad
\begin{aligned}
&\| G_s(t,s_1)x - G_s(t,s_0)x \| \leqq \gamma_3(\sigma)(s_1-s_0)^{\alpha\sigma}(t-s_1)^{-1-\alpha\sigma} \| x \|,\\
&\qquad\qquad 0 \leqq s_0 < s_1 < t \leqq T, \quad x \in X_\theta.
\end{aligned}
$$

For every $\theta \in]0, 1[$ *there is* $\gamma_{4,\theta} > 0$ *such that*

$$
(2.34)\quad
\begin{aligned}
&\| G_s(t,s_0)x - G_s(t,s_1)x \|\\
&\leqq \gamma_{4,\theta}\left(\int_{t-s_1}^{t-s_0} \sigma^{\theta-2}d\sigma + (t-s_1)^{-\max\{\alpha, 1-\theta\}}\right)(s_1-s_0)^{\alpha} \| x \|_\theta,\\
&\qquad\qquad 0 \leqq s_0 < s_1 < t \leqq T, \quad x \in X_\theta.
\end{aligned}
$$

Finally, there is $\gamma_5 > 0$ *such that*

$$
(2.35)\quad
\begin{aligned}
&\| G_s(t,s_1)x - G_s(t,s_0)x \| \leqq \gamma_5(s_1-s_0)^{\alpha}(t-s_1)^{-\alpha} \| x \|_D,\\
&\qquad\qquad 0 \leqq s_0 < s_1 < t \leqq T, \quad x \in D.
\end{aligned}
$$

PROOF. (2.28)(i) is a simple consequence of (2.4), and (2.28)(ii) follows from the equality $G(t,s) = G(t,r)G(r,s)$, $s < r < t$. (2.28)(ii) implies obviously that $G_s(t,s)x$ is differentiable w.r.t. t for $t > s$, and (2.29) holds. Concerning (2.30), it is easy to show (see [14, prop. 3.6(iv)]) that for every $x \in D$ with $A(s)x \in \bar{D}$, $s \to G(t,s)x$ is differentiable in $[0, t]$, and (2.30) holds for $s \leqq t \leqq T$. In the general case, set

$$
v(t) = G_s(t,s)x + G(t,s)A(s)x, \qquad s \leqq t \leqq T.
$$

Then v is differentiable for $t > s$, with $v'(t) = A(t)v(t)$ by (2.29). Moreover

$$
\begin{aligned}
v(t) &= [G(t,s) - e^{(t-s)A(s)}]A(s)x + \partial/\partial s(e^{\sigma A(s)}x)_{|\sigma=t-s} + W_s(t,s)x\\
&= W(t,s)A(s)x + \partial/\partial s(e^{\sigma A(s)}x)_{|\sigma=t-s} + W_s(t,s)x
\end{aligned}
$$

by (2.4) and (2.28)(i), so that v is continuous in $[s, T]$, and (since x belongs to D) $v(s) = 0$. Therefore, v is the classical solution of

$$
v'(t) = A(t)v(t), \quad s < t \leqq T, \quad v(s) = 0.
$$

By uniqueness, $v(t) = 0$ for every $t \in [s, T]$, and (2.30) holds.

Estimates (2.8), (2.9), (2.12), (2.15)(i) imply (2.31)(i), whereas (2.31)(iii) is an obvious consequence of (2.30) and of the boundedness of $\| G(t,s) \|_{L(X)}$. (2.31)(ii) follows now from (2.31)(i) and (2.31)(ii) by interpolation.

From equalities (2.28)(ii) and (2.29) we get $G_{st}(t,s)x = A(t)G(t,r)G_s(r,s)x$, $0 \leqq s < r < t < T$. Using estimates (2.31) and estimates (2.11) of [14], and taking then $r = (t+s)/2$, we get estimates (2.32).

(2.33) is a consequence of (2.8), (2.12), and (2.15)(ii), whereas (2.34) follows from (2.9), (2.12), (2.14), and (2.16). Let us show finally (2.35): for each $x \in D$ we have, by (2.31)(i),

$$\begin{aligned} &\| G_s(t,s_1)x - G_s(t,s_0)x \| \\ &\leqq \| [G(t,s_1) - G(t,s_0)]A(s_1)x \| + \| G(t,s_0)[A(s_1) - A(s_0)]x \| \\ &\leqq \gamma_1 \int_{s_0}^{s_1} \sigma^{-1} d\sigma \, \| A(s_0)x \| + \| G(t,s_0) \|_{L(X)} \| A' \|_\infty (s_1 - s_0) \| x \| \end{aligned}$$

and (2.35) follows easily. ∎

With the aid of estimates (2.31) we can show an integration by parts formula, which will be used in the next section.

Corollary 2.6. *Let $0 \leqq a < b \leqq T$, and let $f \in C^1([a,b];X)$. Then*

$$\begin{aligned} \int_a^t G(t,s) f'(s) ds = &- \int_a^t G_s(t,s)[f(s) - f(t)] ds \\ &- G(t,a)[f(a) - f(t)], \quad a \leqq t \leqq b. \end{aligned} \tag{2.36}$$

If, in addition, f is bounded in $[a,b]$ with values in X_θ for some $\theta \in]0,1[$, then we have also

$$\int_a^t G(t,s) f'(s) ds = - \int_a^t G_s(t,s) f(s) ds + f(t) - G(t,a) f(a), \quad a \leqq t \leqq b \tag{2.37}$$

Proof. For $a < t < b$ and for each $\varepsilon \in]0, t-a[$, the function $s \to G(t,s)[f(s) - f(t)]$ is continuously differentiable in $[a, t-\varepsilon]$, and we have:

$$\partial/\partial s \{ G(t,s)[f(s) - f(t)] \} = G_s(t,s)[f(s) - f(t)] + G(t,s) f'(s).$$

Integrating between a and $t - \varepsilon$ we find

$$\int_a^{t-\varepsilon} G(t,s)f'(s)ds = -\int_a^{t-\varepsilon} G_s(t,s)[f(s)-f(t)]ds$$

(2.38)
$$+ G(t,t-\varepsilon)[f(t-\varepsilon)-f(t)] - G(t,a)[f(a)-f(t)].$$

Letting $\varepsilon \to 0$ in (2.38) and recalling estimate (2.31)(i), we find (2.36). If, in addition, f is bounded with values in X_θ, we can repeat the above procedure with the function $\partial/\partial s[G(t,s)f(s)]$ replacing $\partial/\partial s\{G(t,s)[f(s)-f(t)]\}$. Since X_θ is contained in the closure of D, $G(t,t-\varepsilon)f(t)$ goes to $f(t)$ as $\varepsilon \to 0$ due to Proposition 3.6(ii) of [14], and (2.37) follows easily from estimate (2.31)(ii). ■

In view of Corollary 2.6, we are interested in the regularity properties of the function

(2.39)
$$u(t) = \int_a^t G_s(t,s)f(s)ds, \qquad a \leqq t \leqq b.$$

Using estimates (2.31) and (2.32), many regularity properties of u could be stated; for the sake of brevity we only give a result which will be used in the next section.

PROPOSITION 2.7. *Let f belong to $C^\beta([a,b];X_\theta)$ with $\theta+\beta>1$, and $f(a)=0$. Then the function u defined in* (2.39) *belongs to $C^{\beta+\theta}([a,b];X)$, $u-f$ belongs to $C^{\beta+\theta-1}([a,b];D)$, and*

(2.40)
$$u'(t) = A(t)[u(t)-f(t)], \qquad a \leqq t \leqq b.$$

Moreover, there is $C_7(\theta,\beta)>0$ such that

(2.41)
$$\| u \|_{C^{\theta+\beta}([a,b];X)} + \| u-f \|_{C^{\theta+\beta-1}([a,b];D)} \leqq C_7(\theta,\beta) \| f \|_{C^\beta([a,b];X_\theta)}.$$

PROOF. First of all, we show that $u(t)-f(t)$ belongs to D for every t, and $t \to A(t)[u(t)-f(t)]$ is $(\theta+\beta-1)$-Hölder continuous. By Corollary 2.6 we have

(2.42)
$$u(t)-f(t) = \int_a^t G_s(t,s)[f(s)-f(t)]ds - G(t,a)f(t), \qquad a \leqq t \leqq b.$$

By estimate (2.32)(ii) and assumption (2.3), we get

$$\| G_s(t,s)[f(s)-f(t)] \|_D \leqq \text{const}\cdot(t-s)^{\theta+\beta-2},$$

so that $u(t)-f(t)$ belongs to D for every $t \in [a,b]$. Moreover, using equality (2.4), we get

$$A(t_1)G(t_1-r)x - A(t_0)G(t_0,r)x$$
$$= W_t(t_1,r)x - W_t(t_0,r)x + A(r)[e^{(t_1-r)A(r)}x - e^{(t_0-r)A(r)}x]$$

for $0 \leqq r < t_0 < t_1 \leqq T$, so that, by (2.17) and (2.8), there is $k_1(\theta,\beta) > 0$ such that

$$\begin{aligned}&\| A(t)_1 G(t_1,r)x - A(t_0)G(t_0,r)x \| \\ &\leqq k_1(\theta,\beta)\left[(t_1-t_0)^{\theta+\beta-1}(t_0-r)^{1-\theta-\beta} + \int_{t_0-r}^{t_1-r} \sigma^{-2}d\sigma\right] \| x \|\end{aligned} \tag{2.43}$$

for $0 \leqq r < t_0 < t_1 \leqq T$ and $x \in X$, and

$$\begin{aligned}&\| A(t_1)G(t_1,r)x - A(t_0)G(t_0,r)x \| \\ &\leqq k_2(\theta,\beta)\left[(t_1-t_0)^{\theta+\beta-1}(t_0-r)^{1-\theta-\beta} + \int_{t_0-r}^{t_1-r} \sigma^{\theta-2}d\sigma\right] \| x \|_{X_\theta}\end{aligned} \tag{2.44}$$

for $0 \leqq r < t_0 < t_1 \leqq T$ and $x \in X_\theta$. By equality (2.28), we have

$$A(t_1)G_s(t_1-s)x - A(t_0)G_s(t_0,s)x$$
$$= [A(t_1)G(t_1,r) - A(t_0)G(t_0,r)]G_s(r,s)x$$

for $0 \leqq s < r < t_0 < t_1 \leqq T$. Taking $r = (t_0+s)/2$ and using (2.43), (2.31)(ii) we get

$$\begin{aligned}&\| A(t_1)G_s(t_1,s)x - A(t_0)G_s(t_0,s)x \| \\ &\leqq k_1(\theta,\beta)\gamma_1(\theta)[(t_1-t_0)^{\theta+\beta-1}(t_0-s)^{-\beta} \\ &\qquad + (t_1-t_0)(t_0-s)^{\theta-2}(2t_1-t_0-s)^{-1}] \| x \|_\theta, \\ &\qquad\qquad 0 \leqq s < t_0 < t_1 \leqq T, \quad x \in X_\theta.\end{aligned} \tag{2.45}$$

Using now estimates (2.32)(ii) and (2.44), (2.45), together with estimate (2.11) of [14], we find

$$\begin{aligned}&\| A(t_1)[u(t_1) - f(t_1)] - A(t_0)[u(t_0) - f(t_0)] \| \\ &\leqq \left\| \int_a^{t_0} [A(t_1)G_s(t_1,s) - A(t_0)G_s(t_0,s)][f(s) - f(t_0)]ds \right\| \\ &\quad + \left\| \int_{t_0}^{t_1} A(t_1)G_s(t_1,s)[f(s) - f(t_1)]ds \right\| + \| A(t_1)G(t_1,t_0)[f(t_0) - f(t_1)] \| \\ &\quad + \| [A(t_0)G(t_0,a) - A(t_1)G(t_1,a)]f(t_0) \|\end{aligned}$$

$$\leq \left\{k_1(\theta,\beta)\gamma_1(\theta)\int_a^{t_0}[(t_1-t_0)^{\theta+\beta-1}+(t_1-t_0)(t_0-s)^{\theta+\beta-2}(2t_1-t_0-s)^{-1}]ds\right.$$

$$+\gamma_2(\theta)\int_{t_0}^{t_1}(t_1-s)^{\theta+\beta-2}ds+c(\theta)(t_1-t_0)^{\theta+\beta-1}$$

$$\left.+k_2(\theta,\beta)\left[(t_1-t_0)^{\theta+\beta-1}+\int_{t_0-a}^{t_1-a}s^{\theta+\beta-2}ds\right]\right\}[f]_{C^\beta([a,b];X_\theta)}$$

$$\leq \left\{k_1(\theta,\beta)\gamma_1(\theta)\left[b-a+\int_0^{+\infty}\sigma^{\theta+\beta-2}(2+\sigma)^{-1}d\sigma\right]+\gamma_1(\theta)(\theta+\beta-1)^{-1}\right.$$

$$\left.+c(\theta)+k_2(\theta,\beta)[(b-a)^{1-\theta}+(\theta+\beta-1)^{-1}]\right\}(t_1-t_0)^{\theta+\beta-1}[f]_{C^\beta([a,b];X_\theta)}.$$

Let us show now that u is differentiable and (2.40) holds. We have, for $a \leq t < t+h \leq b$:

$$\left\|\frac{u(t+h)-u(t)}{h}-A(t)[u(t)-f(t)]\right\|$$

$$\leq \left\|\int_a^t\left[\frac{G_s(t+h,s)-G_s(t,s)}{h}-A(t)G_s(t,s)\right][f(s)-f(t)]ds\right\|$$

$$+\left\|\frac{1}{h}\int_t^{t+h}G_s(t+h,s)[f(s)-f(t+h)]ds\right\|$$

$$+\left\|\int_a^t\frac{G_s(t+h,s)-G_s(t,s)}{h}f(t)ds\right.$$

$$\left.+\int_t^{t+h}\frac{G_s(t+h,s)}{h}f(t+h)ds+A(t)G(t,a)f(t)\right\|$$

$$\leq \left\|\int_a^t\int_0^1[A(t+\sigma h)G_s(t+\sigma h,s)-A(t)G_s(t,s)]d\sigma[f(s)-f(t)]ds\right\|$$

$$+\left\|\int_t^{t+h}\frac{G_s(t+h,s)}{h}[f(s)-f(t+h)]ds\right\|$$

$$+\|h^{-1}[G(t+h,t)-1][f(t)-f(t+h)]\|$$

$$+\|\{h^{-1}[G(t+h,a)-G(t,a)]-A(t)G(t,a)\}f(t)\|$$

$$=I_1(h)+I_2(h)+I_3(h)+I_4(h).$$

Thanks to (2.45) we have

$$I_1(h) \leqq k_2(\theta,\beta)\int_a^t \left[s^{\theta+\beta-1}h^{\theta+\beta-1} + \frac{sh}{(t-s)^{2-\theta-\beta}(2h+t-s)}\right] ds [f]_{C^\beta([a,b];X_\theta)}$$

$$\leqq k_2(\theta,\beta)h^{\theta+\beta-1}\left[b-a+\int_0^{+\infty}\sigma^{\theta+\beta-2}(2+\sigma)^{-1}d\sigma\right][f]_{C^\beta([a,b];X_\theta)}.$$

Using (2.32)(ii) we get

$$I_2(h) \leqq \gamma_2(\theta)h^{-1}\int_t^{t+h}(t+h-s)^{\theta+\beta-1}ds[f]_{C^\beta([a,b];X_\theta)}$$

$$\leqq \gamma_2(\theta)(\theta+\beta)^{-1}h^{\theta+\beta-1}[f]_{C^\beta([a,b];X_\theta)}.$$

Thanks to Proposition 2.6(iv) of [14] we have $\| G(t,s)x - x \| \leqq c(\theta)(t-s)^\theta \| x \|_\theta$ for $t > s$ and $x \in X_\theta$, so that

$$I_3(h) \leqq c(\theta)h^{\theta+\beta-1}[f]_{C^\beta([a,b];X_\theta)}.$$

Finally, for $t = a$ we have $I_4(h) = 0$ for every h, whereas for $t > a$ we have obviously $\lim_{h\to 0} I_4(h) = 0$.

Therefore, $A(t)[u(t) - f(t)]$ is the right derivative of $u(t)$ for each $t \in [a, b[$. Since both u and $A(\cdot)[u(\cdot) - f(\cdot)]$ are continuous in $[a, b[$, then u is differentiable in $[a, b[$ and (2.40) hold for $a \leqq t < b$. Since u' is uniformly continuous in $[a, b[$, then u is differentiable also at $t = b$, and (2.40) holds. ■

REMARK 2.8. Assumption $f(a) = 0$ in Proposition 2.7 was made in order to prove regularity of u up to $t = a$. One can easily see that, if $f(a) \neq 0$, then u belongs to $C^{\beta+\theta}([a+\varepsilon, b]; X)$, $u + f$ belongs to $C^{\beta+\theta-1}([a+\varepsilon, b]; D)$, and (2.40) holds in $[a+\varepsilon, b]$ for every $\varepsilon \in]0, b-a[$. ■

The following proposition is concerned with further regularity of $G(t,s)$ w.r.t. t, which is much easier to be treated than further regularity w.r.t. s.

PROPOSITION 2.9. *The function $t \to G(t,s)x$ is twice continuously differentiable in $]s, T]$ for every $x \in X$, and there are constants γ_6, γ_7, $\gamma_6(\theta)$, $\gamma_7(\theta)$ such that we have*:

(2.46)
(i) $\| G_{tt}(t,s)x \| \leqq \gamma_6(t-s)^{-2} \| x \|$, $0 \leqq s < t \leqq T$, $x \in X$,
(ii) $\| G_{tt}(t,s)x \| \leqq \gamma_6(\theta)(t-s)^{\theta-2} \| x \|_\theta$, $0 \leqq s < t \leqq T$, $x \in X_\theta$,
(iii) $\| G_{tt}(t,s)x \| \leqq \gamma_6(t-s)^{-1} \| x \|_D$, $0 \leqq s < t \leqq T$, $x \in D$;

$$
(2.47)\quad
\begin{aligned}
&\text{(i)}\ \| G_{tt}(t_1, s)x - G_{tt}(t_0, s)x \| \leqq \gamma_7 (t_1 - t_0)^{\alpha}(t_0 - s)^{-2-\alpha} \| x \|,\\
&\qquad 0 \leqq s < t_0 < t_1 \leqq T,\ x \in X,\\
&\text{(ii)}\ \| G_{tt}(t_1, s)x - G_{tt}(t_0, s)x \| \leqq \gamma_7(\theta)(t_1 - t_0)^{\alpha}(t_0 - s)^{-2-\alpha+\theta} \| x \|\ \| x \|_{\theta},\\
&\qquad 0 \leqq s < t_0 < t_1 \leqq T,\ x \in X_{\theta},\\
&\text{(iii)}\ \| G_{tt}(t_1, s)x - G_{tt}(t_0, s)x \| \leqq \gamma_7 (t_1 - t_0)^{\alpha}(t_0 - s)^{-1-\alpha} \| x \|_D,\\
&\qquad 0 \leqq s < t_0 < t_1 \leqq T,\ x \in D.
\end{aligned}
$$

PROOF. Let $0 \leqq s < T$ and $\varepsilon \in]0, T - s[$. Consider the problem obtained differentiating formally (1.1) w.r.t. time in $[s + \varepsilon, T]$ with $f = 0$):

$$
(2.48)\quad
\begin{aligned}
&v'(t) = A(t)v(t) + A'(t)G(t, s)x,\\
&s + \varepsilon < t \leqq T;\quad v(s + \varepsilon) = A(s + \varepsilon)G(s + \varepsilon, s)x.
\end{aligned}
$$

It is easy to check that $t \to A'(t)G(t, s)x$ belongs to $C^{\alpha}([s + \varepsilon, T]; X)$ and $v(s + \varepsilon)$ belongs to the closure of D. Therefore (2.48) has a unique solution v, and, by estimates (3.6)(a) and (2.10) of [14], we have:

$$
(2.49)\quad
\begin{aligned}
\| v'(t) \| + \| v(t) \|_D &\leqq \text{const} \cdot (\| A'(\cdot)G(\cdot, s)x \|_{C^{\alpha}([s+\varepsilon, T]; X)}\\
&\quad + (t - s - \varepsilon)^{-1} \| A(s + \varepsilon)G(s + \varepsilon, s)x \|),\\
&\qquad s + \varepsilon < t \leqq T.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\| v'(t) \| \leqq \text{const} \cdot (t - s - \varepsilon)^{-1}\varepsilon^{-1} \| x \| \qquad \text{if } x \in X,\\
&\| v'(t) \| \leqq \text{const} \cdot (t - s - \varepsilon)^{-1}\varepsilon^{\theta-1} \| x \|_{\theta} \qquad \text{if } x \in X_{\theta},\\
&\| v'(t) \| \leqq \text{const} \cdot (t - s - \varepsilon)^{-1} \| x \|_D \qquad \text{if } x \in D.
\end{aligned}
$$

Thanks to estimates (3.6)(a) and (2.12) of [14], we have also

$$
(2.50)\quad
\begin{aligned}
&\| v'(t) - v'(r) \| + \| v(t) - v(r) \|_D\\
&\leqq \text{const} \cdot [(r - s - \varepsilon)^{-1} \| A'(\cdot)G(\cdot, s)x \|_{C^{\alpha}([s+\varepsilon, T]; X)}\\
&\quad + (t - s - \varepsilon)^{1-\alpha} \| A(s + \varepsilon)G(s + \varepsilon, s)x \|](t - r)^{\alpha},\\
&\qquad s + \varepsilon < r < t \leqq T.
\end{aligned}
$$

Now, it is not difficult to prove that to prove that $v(t) = G_t(t, s)x = A(t)G(t, s)x$, we have to write down the i.v. problem satisfied by $v_h(t) = h^{-1}[G(t + h, s)x - G(t, s)x]$ in $[s + \varepsilon, T]$; this lets one check that $v_h \to v$ uniformly in $[s + \varepsilon, T]$. Estimates (2.46) and (2.47) follow now easily. ■

3. A representation formula in nonhomogeneous I.B.V. problems

We consider here problem (1.3), under the ellipticity condition

$$a_{ij}(t,x)\xi_i\xi_j \geqq \nu|\xi|^2, \quad 0\leqq t\leqq T, \quad x\in\bar{\Omega}, \quad \xi=(\xi_1,\dots,\xi_n)\in\mathbf{R}^n; \tag{3.1}$$

Ω is a bounded open set in $\mathbf{R}^n$, with C^2 boundary $\partial\Omega$. The coefficients of the operator

$$\mathscr{A}(t)=a_{ij}(t,\cdot)D_{ij}+b_i(t,\cdot)D_i+c(t,\cdot), \qquad 0\leqq t\leqq T \tag{3.2}$$

satisfy the following regularity assumptions:

(3.3) for every $i,j=1,\dots,n$, a_{ij}, b_i, c are $C^{1+\alpha}$ with respect to time, a_{ij} is C^2 w.r.t. x, b_i is C^1 w.r.t. x, c is continuous w.r.t. x, and we have:

$$\sup_{x\in\bar{\Omega}}\|a_{ij}(\cdot,x)\|_{C^{1+\alpha}([0,T])}+\sup_{x\in\bar{\Omega}}\|b_i(\cdot,x)\|_{C^{1+\alpha}([0,T])}+\sup_{x\in\bar{\Omega}}\|c(\cdot,x)\|_{C^{1+\alpha}([0,T])}<+\infty,$$

$$\sup_{0\leqq t\leqq T}\|a_{ij}(t,\cdot)\|_{C^2(\bar{\Omega})}+\sup_{0\leqq t\leqq T}\|b_i(t,\cdot)\|_{C^1(\bar{\Omega})}+\sup_{0\leqq t\leqq T}\|c(t,\cdot)\|_{C(\bar{\Omega})}<+\infty.$$

3.1. The Dirichlet boundary condition

Under assumptions (3.1), (3.2), (3.3), we consider problem

$$\begin{aligned}&u_t(t,x)=(\mathscr{A}(t)u(t))(x),\ 0\leqq t\leqq T,\ x\in\Omega,\\&u(0,x)=u_0(x),\ x\in\Omega,\\&u(t,x)=g(t,x),\ 0\leqq t\leqq T,\ x\in\partial\Omega.\end{aligned} \tag{3.4}$$

We fix $p\in]1,+\infty[$, and we choose

$$X=L^p(\Omega), \qquad D=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega). \tag{3.5}$$

Then the family of operators

$$A(t):D\to X, \quad A(t)f=\mathscr{A}(t)f, \quad 0\leqq t\leqq T \tag{3.6}$$

satisfies assumptions (2.1), (2.2), (2.3), thanks to [4], [3]. Therefore there exists the evolution operator $G(t,s)$ associated to the family $\{A(t)\}$, and $s\to G(t,s)$ is differentiable for $t>s$ with values in $L(X,D)$. We assume, for simplicity,

$$0\in\rho(A(t)) \qquad \text{for each } t\in[0,T] \tag{3.7}$$

and we define the Dirichlet mapping $D(s):W^{2-1/p,p}(\partial\Omega)\to W^{2,p}(\Omega)$ by $D(s)\psi=v$, where v is the solution of (see [4])

$$\mathscr{A}(s)v = 0 \quad \text{in } \Omega, \qquad v_{|\partial\Omega} = \psi. \tag{3.8}$$

Then $D(s)$ belongs also to $L(L^p(\partial\Omega), W^{1/p-\varepsilon,p}(\Omega))$ for every $\varepsilon > 0$. This was shown in [12, th. 10.1] in the case where $\partial\Omega$ and the coefficients of $\mathscr{A}(s)$ are of class C^∞. But one can see, repeating the proof of Th. 10.1 of [12], that $\partial\Omega$ of class C^2 and assumptions (3.3) are sufficient. Now, recalling again assumption (3.3), it is easy to see that

$$\begin{aligned} s \to D(s) \in C^1([0, T]; L(L^p(\partial\Omega), W^{1/p-\varepsilon,p}(\Omega)) \\ \cap C([0, T]; L(W^{2-1/p,p}(\partial\Omega), W^{2,p}(\Omega)). \end{aligned} \tag{3.9}$$

Proposition 3.1. *Under the previous assumptions and notation, for every* $u_0 \in W^{2,p}(\Omega)$, $g: [0, T] \times \partial\Omega \to \mathbf{R}$ *s.t.* $t \to g(t, \cdot) \in C^1([0, T], L^p(\partial\Omega)) \cap C([0, T], W^{2-1/p,p}(\partial\Omega))$ *and* $u_{0|\partial\Omega} = g(0, \cdot)$, *problem* (3.4) *has a unique solution* u, *such that* $t \to u(t, \cdot)$ *belongs to* $C^1([0, T], L^p(\Omega)) \cap C([0, T], W^{2,p}(\Omega))$, *and* u *is given by formula* (1.4). *We have also*

$$\begin{aligned} u(t, \cdot) = G(t, 0)(u_0 - D(0)g(0, \cdot)) \\ - \int_0^t G(t, s)[d/ds(D(s)g(s, \cdot))]ds + D(t)g(t, \cdot), \\ 0 \leqq t \leqq T. \end{aligned} \tag{3.10}$$

Proof. Uniqueness of the solution to (3.4) follows obviously from uniqueness in the homogenous problem. Let us show that the function u defined in (3.10) is the solution of (3.4). Since $g(0, \cdot)$ belongs to $W^{2-1/p,p}(\partial\Omega)$, and $D(0)g(0, \cdot)$ belongs to $W^{2,p}(\Omega)$ and, due to the compatibility condition $u_{0|\partial\Omega} = g(0, \cdot)$, we have $u_0 - D(0)g(0, \cdot) \in D$. Moreover, for every $\theta \in]0, 1/2p[$ the interpolation space X_θ coincides algebraically and topologically with the Besov space $B_\infty^{2\theta,p}(\Omega)$ thanks to [8]. Since $W^{1/p-\varepsilon,p}(\Omega)$ is continuously embedded in $B_\infty^{1/p-\varepsilon}(\Omega)$ for each $\varepsilon \in]0, 1/p[$ (see, e.g., [18, th. 4.6.1 p. 327]), then, due to (3.9), $t \to d/dt\ (D(t)g(t, \cdot))$ belongs to $C([0, T], X_\theta)$ for each $\theta \in]0, 1/2p[$. Therefore, by Proposition 2.6(v) and Proposition 3.5(ii) of [14], the function

$$v(t) = u(t, \cdot) - D(t)g(t, \cdot), \qquad 0 \leqq t \leqq T$$

belongs to $C^1([0, T], X) \cap C([0, T], D)$ and satisfies

$$v'(t) = A(t)v(t) - d/dt(D(t)g(t, \cdot)), \quad 0 \leqq t \leqq T; \quad v(0) = u_0 - D(0)g(0, \cdot).$$

Using again (3.9), we find that $t \to D(t)g(t, \cdot)$ belongs to $C([0, T], W^{2,p}(\Omega))$. Summing up, we get $t \to u(t, \cdot) \in C^1([0, T], L^p(\Omega)) \cap C([0, T], W^{2,p}(\Omega))$, and

u satisfies (3.4). The representation formula (1.4) for u is obtained integrating by parts in (3.10); this can be done thanks to Corollary 2.6. ■

3.2. The mixed boundary condition

Under assumptions (3.1), (3.2), (3.3), we consider now problem

$$(3.11)\qquad \begin{aligned} &u_t(t,x) = (\mathscr{A}(t)u(t))(x), \qquad 0 \leqq t \leqq T, \quad x \in \Omega, \\ &u(0,x) = u_0(x), \qquad x \in \Omega, \\ &\mathscr{B}u(t,x) = g(t,x), \qquad 0 \leqq t \leqq T, \quad x \in \partial\Omega, \end{aligned}$$

where

$$(3.12)\qquad \mathscr{B}f(x) = \beta_i(x)D_if(x) + \gamma(x)f(x), \qquad x \in \partial\Omega$$

and

$$(3.13)\qquad \beta_i, \gamma \in C^1(\partial\Omega); \quad \beta_i(x)\nu_i(x) \neq 0 \quad \text{for } x \in \partial\Omega.$$

We repeat the procedure of the previous subsection, setting

$$(3.14)\qquad X = L^p(\Omega), \qquad D = \{f \in W^{2,p}(\Omega); \beta_iD_if + \gamma f = 0 \text{ on } \partial\Omega\}.$$

Again, the family of operators $A(t)$ defined by (3.6) satisfies assumptions (2.1), (2.2), (2.3) thanks to [4], [3]. We assume that (3.7) holds, and we define the mixed mapping $M(s): W^{1,1/p,p}(\partial\Omega) \to W^{2,p}(\Omega)$ by $M(s)\psi = v$, where v is the unique solution of the elliptic problem

$$(3.15)\qquad \mathscr{A}(s)v = 0 \quad \text{in } \Omega, \quad \beta_iD_iv + \gamma v = \psi \quad \text{on } \partial\Omega.$$

Then $M(s)$ belongs also to $L(L^p(\partial\Omega); W^{1+1/p-\varepsilon,p}(\Omega))$ for every $\varepsilon \in]0, 1 + 1/p[$, thanks to [13, th. 4.1]. In fact, in [13] it is assumed that $\partial\Omega$ and the coefficients of $\mathscr{A}(s)$ and $\mathscr{B}$ are of class C^∞, but one can check that the statement of Theorem 4.1 in [13] holds true also under our hypotheses. Now, using assumption (3.3), it is not difficult to see that

$$(3.16)\qquad \begin{aligned} &s \to M(s) \in C^1([0,T]; L(L^p(\partial\Omega), W^{1+1/p-\varepsilon,p}(\Omega)) \\ &\qquad \cap\, C([0,T]; L(W^{1-1/p,p}(\partial\Omega), W^{2,p}(\Omega)). \end{aligned}$$

PROPOSITION 3.2. *Let assumptions* (3.1), (3.3), (3.13), (3.7) *hold. Then for every* $u_0 \in W^{2,p}(\Omega)$, $g: [0,T] \times \partial\Omega \to \mathbf{R}$ *such that*

$$t \to g(t,\cdot) \in C^{1/2}([0,T], L^p(\partial\Omega)) \cap C([0,T], W^{1-1/p,p}(\partial\Omega)) \quad \textit{and} \quad \mathscr{B}u_0 = g(0,\cdot),$$

the function u given by formula (1.6) *belongs to* $C^1([0,T],L^p(\Omega))\cap C([0,T],W^{2,p}(\omega))$ *and it is the unique solution of problem* (3.11).

PROOF. One can get heuristically formula (1.6) arguing as in the proof of Proposition 3.1; but in this case a formula similar to (3.10) does not make sense, since $M(s)g(s,\cdot)$ is not differentiable. However, (1.6) makes sense, because $t\to M(t)g(t,\cdot)$ belongs to $C^{1/2}([0,T],X_\theta)$ for every $\theta\in]0,1/2+1/2p[$: actually, it belongs to $C^{1/2}([0,T],W^{1+1/p-\varepsilon,p}(\Omega))$ thanks to (3.16), and $W^{1+1/p-\varepsilon,p}(\Omega)$ is continuously embedded in the Besov space $B_\infty^{1+1/p-\varepsilon,p}(\Omega)$ (see, e.g., [18, th. 4.6.1 p. 327]), which coincides algebraically and topologically with X_θ for $\theta=1/2+1/2p-\varepsilon/2$ due to [8]. Therefore formula (1.6) makes sense, thanks to estimate (2.32)(ii). To show the statement, set $u(t,\cdot)=u_1(t)+u_2(t)$, where

$$u_1(t)=G(t,0)(u_0-M(0)g(0,\cdot))+M(0)g(0,\cdot),\quad 0\leqq t\leqq T,$$

$$u_2(t)=\int_0^t G_s(t,s)(M(s)g(s,\cdot)-M(0)g(0,\cdot))ds,\quad 0\leqq t\leqq T.$$

Since $g(0,\cdot)$ belongs to $W^{1-1/p,p}(\partial\Omega)$, then $M(0)g(0,\cdot)$ belongs to $W^{2,p}(\Omega)$ and, due to the compatibility condition $\mathscr{B}u_0=g(0,\cdot)$, we have $u_0-M(0)g(0,\cdot)\in D$; therefore $t\to G(t,0)(u_0-M(0)g(0,\cdot))$ belongs to $C^1([0,T],X)\cap C([0,T],D)$, so that u_1 belongs to $C^1([0,T],L^p(\Omega))\cap C([0,T],W^{2,p}(\Omega))$ and satisfies

$$\begin{aligned} u_1'(t)&=A(t)G(t,0)(u_0-M(0)g(0,\cdot))\\ &=\mathscr{A}(t)u_1(t)-\mathscr{A}(t)M(0)g(0,\cdot),\quad 0\leqq t\leqq T,\\ u_1(0)&=u_0,\\ \mathscr{B}u_1(t)&=g(0,\cdot),\quad 0\leqq t\leqq T. \end{aligned}$$

We remarked before that $t\to M(t)g(t,\cdot)$ belongs to $C^{1/2}([0,T],X_\theta)$ for every $\theta\in]0,1/2+1/2p[$; choosing $\theta\in]1/2,1/2+1/2p[$ and applying Proposition 2.7, we find that u_2 belongs to $C^1([0,T],L^p(\Omega))$, $t\to u_2(t)-M(t)g(t,\cdot)+M(0)g(0,\cdot)$ belongs to $C([0,T],D)$ (so that, due to (3.16), u_2 belongs to $C([0,T],W^{2,p}(\Omega))$), and

$$
\begin{aligned}
u_2'(t) &= A(t)[u_2(t) - M(t)g(t,\cdot) + M(0)g(0,\cdot)] \\
&= \mathscr{A}(t)u_2(t) + \mathscr{A}(t)M(0)g(0,\cdot), \quad 0 \leqq t \leqq T, \\
u_2(0) &= 0, \\
\mathscr{B}u_2(t) &= -\mathscr{B}[-M(t)g(t,\cdot) + M(0)g(0,\cdot)] = g(t,\cdot) - g(0,\cdot), \quad 0 \leqq t \leqq T.
\end{aligned}
$$

Summing up, we find that u belongs to $C^1([0, T], L^p(\Omega)) \cap C([0, T], W^{2,p}(\Omega))$ and satisfies (3.11). Finally, uniqueness of the solution to (3.11) is an obvious consequence of uniqueness in the homogeneous case. ■

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